Math Circles - Intro to Combinatorics - Winter 2024

Lecture 2

February 14th, 2024

1 Expanding Polynomials

In many branches of math polynomials are important. We might care about the roots of a polynomial or some other component of its structure such as the coefficients of various terms. In Combinatorics especially we often use polynomials to count objects and the coefficients provide the count while the power of the term tells us what object we are counting. So lets look at expanding polynomials.

Example 1.0.1 *Consider* $(x+y)^2$. *We can expand this by had using the foil method. Thus* $(x+y)(x+y) = x(x+y)+y(x+y) = x^2+xy+yx+y^2 = x^2+2xy+y^2$.

That wasn't so bad, but what if we want to expand something with a larger exponent?
Consider

$$\begin{aligned}(x+y)^4 &= (x+y)(x+y)(x+y)(x+y) = (x^2+2xy+y^2)(x+y)(x+y)\\ &= (x^2+2xy+y^2)(x^2+2xy+y^2) = x^2(x^2+2xy+y^2)+2xy(x^2+2xy+y^2)+y^2(x^2+2xy+y^2)\\ &= x^4+2x^3y+x^2y^2+2x^3y+4x^2y^2+2xy^2+y^2x^2+2xy^3+y^4\\ &= x^4+4x^3y+6x^2y^2+4xy^3+y^4\end{aligned}$$

That was doable, but not very quick. We can in theory keep doing this process of foiling with larger exponents, but with something like $(x+y)^{27}$ *it would take quite awhile. So there must be a better way to calculate this.*

2 The Binomial Theorem

There is a better way to calculate these coefficients.

Theorem 2.0.1 *(The Binomial Theorem) For a non-negative integer* n, $(x+y)^n = \sum_{i=0}^{n}\binom{n}{i}x^iy^{n-i}$

Proof 1 *(sketch) The coefficient of* x^iy^{n-i} *is the the number of ways it is possible to pick from the* n $(x+y)$ *terms* i x*'s, which fixes the other terms to have* y *chosen. There are* $\binom{n}{i}$ *ways to do this.*

Let's use this information to try and answer a question.

Corollary 2.0.2 *For any* $n \in \mathbb{N}$, $\sum_{i=0}^{n}\binom{n}{i} = ?$

Problems 2.0.3 *Try some small examples to figure out what the solution to the sum above is. How does the sum relate to the binomial theorem?*

We can start by writing out the binomial expansions using the binomial theorem for $n=1,2,3,$ *and* 4. *We have* $n=1$, $x+y$
$n=2$, $x^2+2xy+y^2$
$n=3$, $x^3+3x^2y+3xy^2+y^3$
$n=4$, $x^4+4x^3y+6x^2y^2+4xy^3+y^4$. *Looking at the sum of the coefficients we get* $2,4,8,$ *and* 16. *What pattern is this?*

Perhaps you have noticed that the sum is 2^n. *In mathematics we always have to prove something is true before using it.*

So how can we prove this? What tools do we have? We have the Binomial Theorem! Notice that $2 = 1+1$, *thus* $2^n = (1+1)^n = \sum_{i=0}^{n} \binom{n}{i} 1^i 1^{n-i} = \sum_{i=0}^{n} \binom{n}{i}$.

It is clear that we can plug numbers into the binomial theorem for x and y. In the following problem we will look at how we can use this theorem to calculate specific coefficients.

Example 2.0.4 *Suppose you want to calculate the coefficient of* x^3 *in* $(x+2)^12$. *We can use the binomial theorem to do this.* $(x+2)^12 = \sum_{i=0}^{1} 2\binom{12}{i} x^i 2^{n-i}$. *Thus* x^3 *has coefficient* $\binom{12}{3} 2^{12-3} = \binom{12}{3} 2^9$.

Problems 2.0.5 *Here are some problems for you to try.*

1. *Find the coefficient of* x^2 *in* $(x+4)^7$.
2. *Find the coefficient of* x^3 *in* $(x-3)^{11}$.
3. *What is* $\binom{11}{3}$ *the coefficient of in* $(x+y)^{11}$*?*
4. *What is the coefficient of* $x^3 y^{n-3}$ *in* $(x-y)^n$*? What must* n *be if the exponent on* y *is also* 3*?*

Solution 2.0.6 *We will use the binomial theorem for all of these problems.*

1. $(x+4)^7 = \sum_{i=0}^{7} \binom{7}{i} x^i 4^{7-i}$ *so* x^2 *has coefficient* $\binom{7}{2} 4^{7-2} = \binom{7}{2} 4^5$
2. $(x-3)^{11} = (x+(-3))^{11} = \sum_{i=0}^{11} \binom{11}{i} x^i (-3)^{11-i}$. *Thus the coefficient of* x^3 *is* $\binom{7}{3}(-3)^{7-3} = \binom{7}{3} 3^4$.
3. *Since we have* $\binom{11}{3}$ *it is either the coefficient of* $x^3 y^{11-3}$ *or the coefficient of* $x^{11-3} y^3$ *since* $\binom{11}{3} = \binom{11}{11-3}$
4. *We can write* $(x-y)^n = (x+(-y))^n = \sum_{i=0}^{n} \binom{n}{i} x^i (-y)^{n-i} = \sum_{i=0}^{n} \binom{n}{i} x^i (-1)^{n-i} y^{n-i}$. *Thus the coefficient we want is* $\binom{n}{3}(-1)^{n-3}$. *If* y *has exponent* 3 *then* $n-3=3$ *so* $n=6$ *and the coefficient would be* $-\binom{6}{3}$.

3 What is all of this used for?

You might be wondering why we care about all of this. The binomial theorem is one of the base tools for a branch of Combinatorics called enumeration. It is heavily used in generating functions. Generating functions often count infinite sets of objects. Where the coefficient is the number of objects and the exponent tells us the type of object. We won't cover any details here, but we will discuss one example of how what we have learned about the binomial series is counting something.

Example 3.0.1 *Consider* $(x+y)^4$. *We have expanded this and know it is* $x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4$. *We also know from the binomial theorem that this can be written as* $\sum_{i=0}^{4} \binom{4}{i} x^i y^{4-i}$. *So what is this counting as a generating function? Imagine that we want to build strings of zeros and ones of length 4. This generating function counts all the ways to do this. Imagine* x *represents one and* y *represents zero. Then the* $x^i y^{4-i}$ *tells us the number of ones and zero in the string, and the coefficient tells us how many different strings we can build. Let's write some of them out.*

Using four one's we have 1111 *which is represented by* x^4.
Using three one's we have 1110, 1101, 1011, *and* 01111. *These are represented by* x^3y.
We can continue this idea to see all strings of length four are counted the coefficients and sorted by type of monomial.

You might be wondering what we do with more complex ideas of it the exponent is a negative or a fraction. We have a variation of the binomial theorem called the negative binomial theorem, and a variation of factorials, called a falling factorial that allow us to use these same techniques. We won't go into any detail here about these concepts, but we provide the names in case you wish to look them up.

4 Coefficient Patterns

As you have been working through problems today, you may have started to notice some patterns in coefficients. We are going to start investigating these patterns. On today's problem sheet we will see a some practice problems from today's topic and then the start of the pattern investigation.